

# Quantitative Economics for the Evaluation of the European Policy

Dipartimento di Economia e Management

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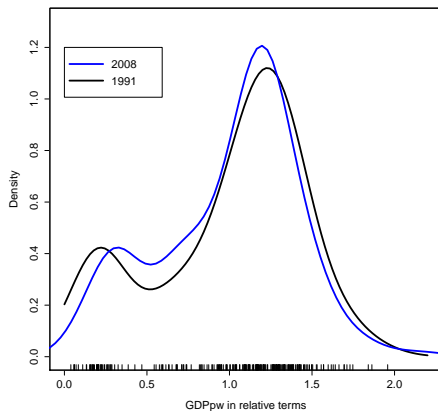
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# Distribution dynamics



**Figura:** Estimated distributions of (relative) GDP per worker in 1991 and 2008 in 254 NUTS-2 European regions.

# Distribution Dynamics (Quah 1993, 1996, 1997)

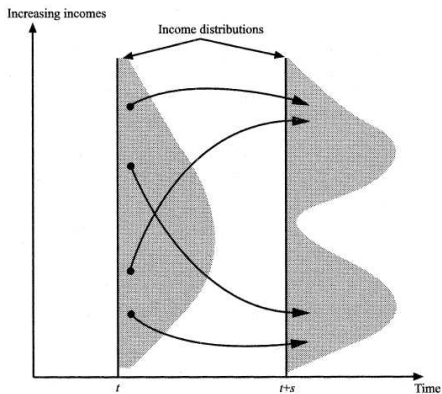


Fig. 1. Twin-peaks distribution dynamics.

# Markov matrix with discrete state space

Define a **set of states** for  $Y/L$ :  $s = \{s_1, s_2, \dots, s_K\}$ .

The **probability** of region  $i$  to transit from state  $k$  to state  $q$  with lag  $\tau$  is defined as:

$$p_{qk} = Pr(Y/L_{i,t+\tau} \in s_q | Y/L_{i,t} \in s_k). \quad (1)$$

If the dynamics of distribution follows a **Markov process**, then the dynamics of the masses of probability related to different states can be represented by:

$$\pi_{t+\tau} = \pi_t P, \quad (2)$$

where  $\pi_t = (\pi_{1,t}, \dots, \pi_{K,t})$  (row vector) and  $\pi_{k,t}$  is the mass of probability of distribution in state  $k$  at period  $t$  and  $P$ , which collect all  $p_{qk}$ , is called the **Markov transition matrix** (dimensions  $K \times K$ ).

Under some regularity conditions there exists an **ergodic (equilibrium) distribution**  $\pi_\infty$  such that:

$$\pi_\infty = \pi_\infty P. \quad (3)$$

## Distribution dynamics (cont.d)

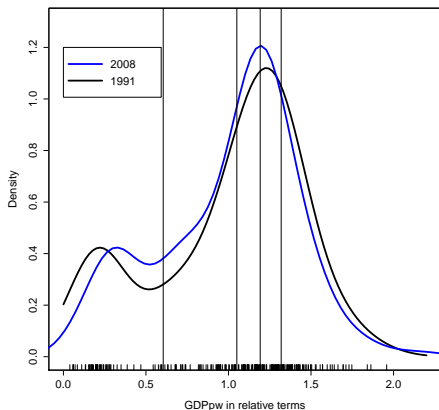


Figura: Estimated distributions of (relative) GDP per worker in 1991 and 2008 in 257 NUTS-2 European regions and state space by the quantiles of distribution

# Estimated Markov matrix with discrete state space

State	1	2	3	4	5
Range	0.06-0.60	0.60-1.05	1.05-1.19	1.19-1.32	1.32-2.22

**Tabella:** Definition of the space states based on the quantile distribution of observations

$t \parallel t + 10$	1	2	3	4	5
1	388.00	31.00	0.00	0.00	0.00
2	16.00	331.00	16.00	6.00	0.00
3	0.00	79.00	214.00	83.00	22.00
4	0.00	12.00	140.00	184.00	106.00
5	0.00	0.00	54.00	107.00	266.00

**Tabella:** Markov matrix for our sample

# Estimated Markov transition matrix

$t \parallel t + 10$	1	2	3	4	5
1	0.93	0.07	0.00	0.00	0.00
2	0.04	0.90	0.04	0.02	0.00
3	0.00	0.20	0.54	0.21	0.06
4	0.00	0.03	0.32	0.42	0.24
5	0.00	0.00	0.13	0.25	0.62

Tabella: Markov transition matrix for our sample

where the **maximum likelihood estimator** of transition probability is given by:

$$\hat{p}_{qk} = \frac{\text{number of observations starting from state } k \text{ and arrived to state } q}{\text{total number of observations starting from state } k}$$

# Estimated ergodic distribution

From the estimate of  $\hat{P}$  we can estimate the ergodic distribution:

$$\hat{\pi}_{\infty} = \hat{\pi}_{\infty} \hat{P};$$

in particular:

State	1	2	3	4	5
Mass of probability	0.26	0.45	0.12	0.09	0.07

Tabella: Ergodic distribution



## Another definition of states spaces ...

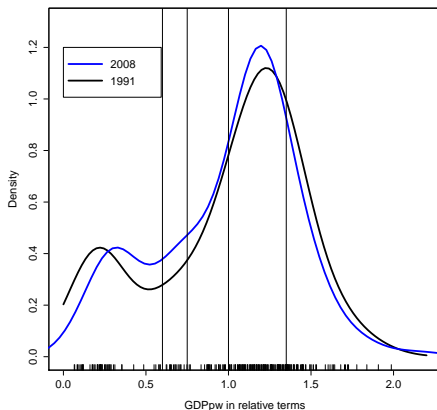


Figura: Estimated distributions of (relative) GDP per worker in 1991 and 2008 in 257 NUTS-2 European regions and alternative state space

## Another definition of states spaces ... (cont.d)

State	1	2	3	4	5
Range	0.06-0.60	0.60-0.75	0.75-1.00	1.00-1.35	1.35-2.22

Tabella: Another definition of the space states

$t \parallel t + 10$	1	2	3	4	5
1	0.93	0.07	0.00	0.00	0.00
2	0.12	0.67	0.20	0.01	0.00
3	0.00	0.14	0.76	0.10	0.00
4	0.00	0.00	0.09	0.79	0.13
5	0.00	0.00	0.00	0.41	0.59

Tabella: Markov transition matrix for our sample

State	1	2	3	4	5
Mass of probability	0.25	0.16	0.23	0.28	0.09

Tabella: Ergodic distribution

# Markov matrix with continuous state space

The definition of the state space may crucially affect the result. Possible solution: **the use of continuous state space**.

Markov matrix with continuous state space becomes a **conditioned distribution**, also denoted **stochastic kernel**:

$$g_{\tau}(Y/L_{i,t+\tau}|Y/L_{i,t}) \equiv \frac{f(Y/L_{i,t+\tau}, Y/L_{i,t})}{r(Y/L_{i,t})}$$

Accordingly the **ergodic distribution** solves:

$$f_{\infty}(x) = \int_0^{\infty} g_{\tau}(x|z) f_{\infty}(z) dz, \quad (4)$$

where  $x$  and  $z$  are two levels of  $Y/L$ ,  $g_{\tau}(x|z)$  is the density of  $x$ , given  $z$ ,  $\tau$  periods ahead, under the constraint

$$\int_0^{\infty} f_{\infty}(x) dx = 1. \quad (5)$$

# Ergodic distribution with normalized variable

Since in our estimates all variables are normalized with respect to their average, the ergodic distribution must respect the additional constraint:

$$\int_0^{\infty} f_{\infty}(x) x dx = 1. \quad (6)$$

Then the true ergodic distribution is:

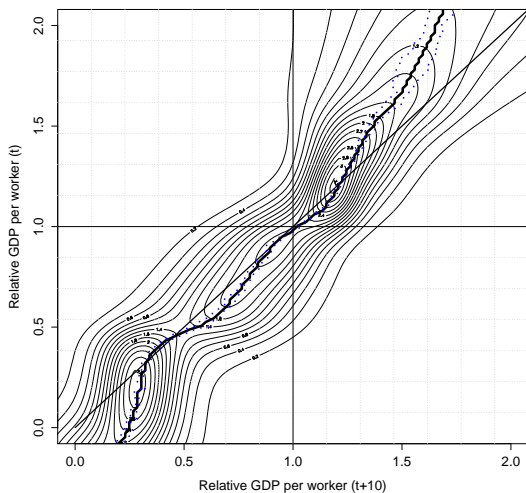
$$f_{\infty}(x) = \tilde{\mu}_x \tilde{f}_{\infty}(x), \quad (7)$$

where:

$$\tilde{\mu}_x = \int_0^{\infty} \tilde{f}_{\infty}(x) x dx \quad (8)$$

and  $\tilde{f}_{\infty}$  satisfies Eqq. (4) and (5).

# Stochastic kernel



## Median of the stochastic kernel

The bold line represents the **median** of the estimated conditioned distribution  $g_\tau(Y/L_{i,t+\tau}|Y/L_{i,t})$ , denoted by  $\text{med}(Y/L_{i,t})$ .

The median is crucial to understand the dynamics of the **mass of distribution**; in particular:

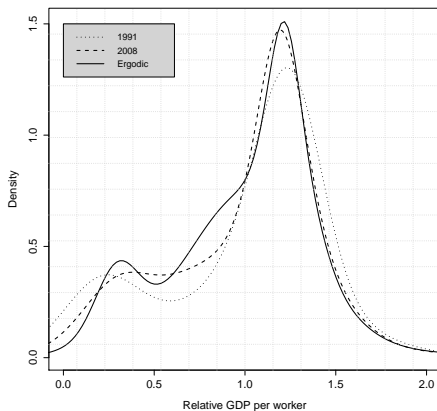
- if  $\text{med}(Y/L_{i,t}) > Y/L_{i,t}$  then we expect that **locally** the mass is shifting ahead around  $Y/L_{i,t}$
- if  $\text{med}(Y/L_{i,t}) < Y/L_{i,t}$  then we expect that **locally** the mass is shifting behind around  $Y/L_{i,t}$
- if  $\text{med}(Y/L_{i,t}) = Y/L_{i,t}$  then we expect that **locally** the mass is stable around  $Y/L_{i,t}$ , i.e.  $Y/L_{i,t}$  is a possible **equilibrium**.

Another perspective to see the estimated stochastic kernel is to think about it how a stochastic difference equation:

$$Y/L_{i,t+\tau} = \phi(Y/L_{i,t}) + \varepsilon_{i,t+\tau} \quad (9)$$

$\Rightarrow$  if around an equilibrium  $\text{med}(Y/L_{i,t})$  crosses from below, this equilibrium is **stable**, if it crosses from above, this equilibrium is **unstable**.

# Estimated ergodic distribution



**Figura:** Estimated ergodic distribution of (relative) GDP per worker in 1991 and 2008 in 257 NUTS-2 European regions