Quantitative Economics for the Evaluation of the European Policy

Dipartimento di Economia e Management

Co-funded by the Erasmus+ Programme of the European Union

Project funded by

European Commission Erasmus + Programme –Jean Monnet Action

Project number 553280-EPP-1-2015-1-IT-EPPJMO-MODULE

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04/10/2016

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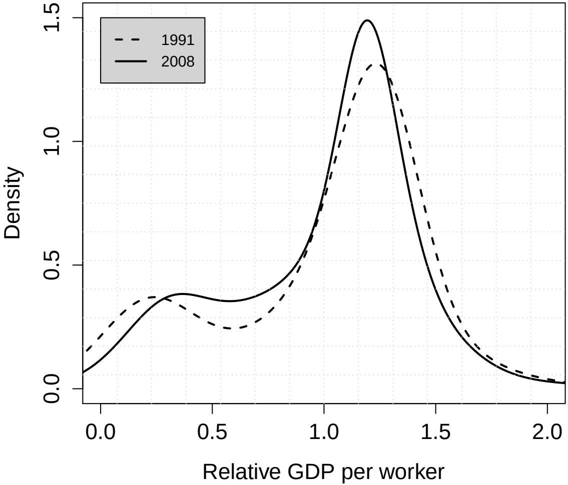
Distribution of Regional GDP per Worker

	1991	2008
Gini	0.25	0.23
BIPOL	0.83	0.78



Estimate of The Density Function

Let be x a continuous random variable and f its probability density function (pdf).

The pdf characterizes the distribution of the random variable x since it tells "how x is distributed".

Moreover, from pdf it is possible to calculate the mean and the variance (it they exists) of x and the probability that x takes on values in a given interval.

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Histograms are nonparametric estimates of an *unknown density function*, $f(x)$, **without assuming any well-known functional form**. In order to build an histogram, you have to:

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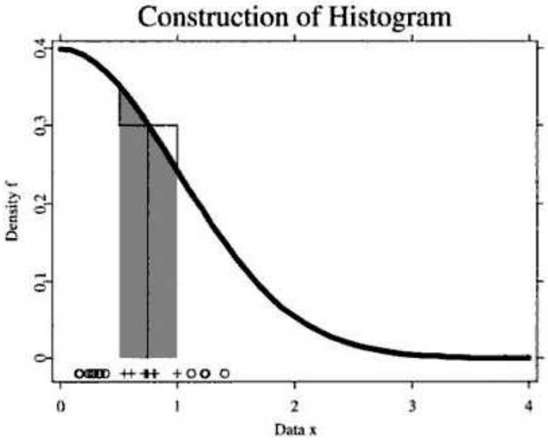
2. count how many observations fall into each bin (n_j for each bin j);
3. for each bin divide the frequency by the sample size n and the binwidth h, to get the relative frequencies $f_j = \frac{n_j}{nh}$

Histogram: Cont.



Histogram: Cont.

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$\rightarrow$ we need to find an "optimal" binwidth, which represents an optimal compromise.

Histogram: Cont.

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1. each observation x in $[m_j - \frac{h}{2}, m_j + \frac{h}{2})$ is estimated by the same value, $\hat{f}_h(m_j)$, where m_j is the center of the bin;
2. $f(x)$ is estimated using the observations that fall in the interval containing x, and that receive the same weight in the estimation. That is, for $x \in B_j$,

$$\hat{f}_h(m_j) = \frac{1}{nh} \sum_{i=1}^{n} I(X_i \in B_j),$$

where I is the indicator function.

Nonparametric density estimation

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- Density estimation is a generalization of the histogram.
- It is based on **Kernel functions**: estimate $f(x)$ using the observations that fall into an interval around x, which (typically) receive decreasing weight the further they are from x.

Kernel functions

Consider the *uniform* kernel function, which assigns *the same weight to all observations in an interval* of length $2h$ around observation x, $[x-h, x+h)$:

$$\hat{f}_h(x) = \frac{1}{2nh} \sharp\{X_i \in [x-h, x+h)\}$$

can be obtained by means of a kernel function $K(u)$ such that:

$$K(u) = \frac{1}{2} I(|u| \leq 1)$$

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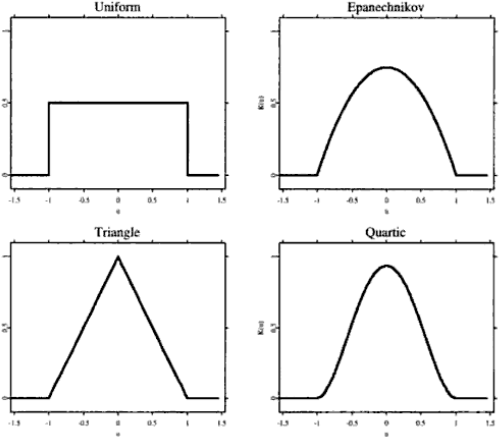
- For each observation that falls into the interval $[x-h, x+h)$ the indicator function takes on value 1
- Each contribution to the function is weighted equally no matter how close the observation X_i is to x

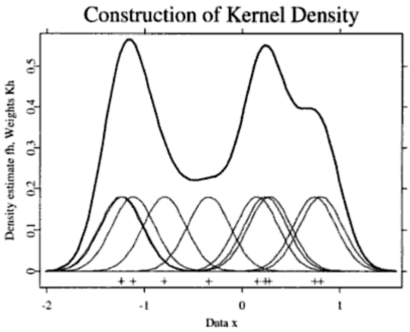
Kernel functions: Cont.

A Kernel function in general (e.g. Epanechnikov, Gaussian, etc), assigns higher weights to observations in $[x - h, x + h)$ closer to x.



Kernel density

A kernel density estimation appears as a sum of bumps: at a given x, the value of $\hat{f}_h(x)$ is found by vertically summing over the "bumps":



$$\hat{f}_h(x) = \sum_{i=1}^{n} \frac{1}{nh} K\left(\frac{x - X_i}{h}\right) = \sum_{i=1}^{n} \frac{1}{n} \underbrace{K_h(x - X_i)}_{\text{"rescaled kernel function"}}$$

Properties of Kernel density estimator

Same problems found for the histogram, that is the bias and the variance depending on h, also hold for the Kernel:

$$Bias\{\hat{f}_h(x)\} = E\{\hat{f}_h(x)\} - f(x);$$

that positively depends on h;

$$Var\{\hat{f}_h(x)\} = Var\left\{\sum_{i=1}^{n} \frac{1}{n} K_h(x - X_i)\right\};$$

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that negatively depends on h.
So, how do we choose h given the trade-off between bias and variance?

Choosing the bandwidth h

(a) Define MSE (mean squared error)

$$MSE\{\hat{f}_h(x)\} = E[\{\hat{f}_h(x) - f(x)\}^2]$$

$$\dots$$

$$MSE\{\hat{f}_h(x)\} = Var\{\hat{f}_h(x)\} + [Bias\{\hat{f}_h(x)\}]^2$$

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(b) Define MISE (mean integrated squared error), global measure:

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(d) One possibility is a plug-in method suggested by Silverman, and consists in assuming that the unknown function is a Gaussian density function (whose variance is estimated by the sample variance). In this case h_{opt} has a simple formulation, and can be defined as a rule-of-thumb bandwidth.

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- In particular, a natural way to deal with long-tailed densities is to use a broader kerne l in regions of low density.
- Thus an observation in the tail would have its mass smudged out over a wider range than one in the main part of the distribution.

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- The adaptive kernel approach copes with this problem by means of a two-stage procedure:
  1. get an initial estimate to have a rough idea of the density
  2. use the former density to get a pattern of bandwidths corresponding to various observations to be used in a second estimate

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where g is the geometric mean of the $\tilde{f}(x_i)$, $\log g = n^{-1} \sum \log \tilde{f}(x_i)$; and α the *sensitivity parameter* $(\alpha \leq 0)$

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Step 3 Define the *adaptive kernel estimate* $\hat{f}(x)$ by:

$$\hat{f}(x) = nh^{-1} \sum \lambda_i^{-1} K\{h^{-1} \lambda_i^{-1} (x - X_i)\} \tag{2}$$

Bootstrap

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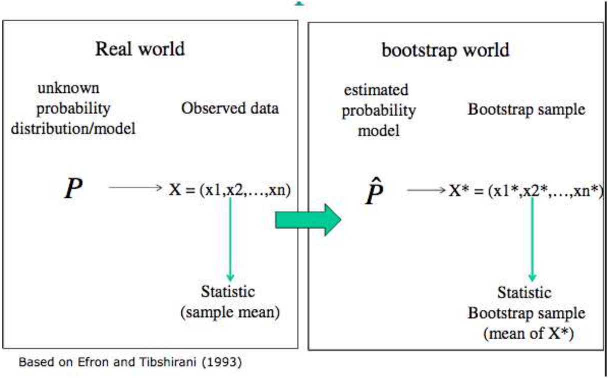

Based on Efron and Tibshirani (1993)

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  2. If the number of resamples (B) from the original sample increases, the bootstrap distribution approaches the sample distribution.

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The distribution of $\hat{f}^*$ about $\hat{f}$ can therefore be used to mimic the distribution of $\hat{f}$ about f, that is it can be used to calculate the confidence intervals for estimates.

References

Histogram and Density Estimation

Bowman, A.W. and Azzalini A. (1997). Applied smoothing techniques for data analysis: the kernel approach with S-Plus illustrations: the kernel approach with S-Plus illustrations. *Oxford University Press*.

- Estimate: Chapter 1
- Inference (confidence bands): Chapter 2

Adaptive Density Estimation

Silverman, B.W. (1986). Density estimation for statistics and data analysis. *CRC press*.

- Estimate: Chapter 5.3